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BellTest and CHSH experiments with more than two settings

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Abstract

Strong nonlocality allows signals faster than light. Weak nonlocality is a statistical property of classical events for which there is no realistic local theory. This requires the violation of at least one general Bell inequality. The theory of ideal quantum measurements predicts weak nonlocality but not strong nonlocality. *Bell experiment* here refers to any experiment designed to demonstrate weak nonlocality. BellTest is a computer program generally available on the Web to help planning and analysis of such Bell experiments. In Mode 1 it obtains general Bell inequalities. In Mode 2 it tests for their violation. We describe its use, with some new results for illustration.

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1. Introduction

Do we live in a completely local world, or is there some kind of nonlocality? There is no evidence for the strong nonlocality that would allow signals faster than the velocity of light. But there is evidence in favour of a kind of *weak nonlocality* which excludes any local hidden variable theory of quantum measurement. The experimental evidence is not conclusive, because every experimental test has at least one loophole.

In the 1935 paper of Einstein, Podolsky and Rosen (EPR) [1], the problems of locality are implicit, giving the impression that they are matters of interpretation, which could not be tested by experiment. With Bell's original inequality [2], weak nonlocality and the existence of local hidden variables became a matter of experiment, apart from the limitations due to the difference between ideal and real quantum measurements. These limitations were overcome, in principle, for a particular type of experiment by Clauser, Horne, Shimony and Holt [3].

Wigner in 1970 [4] expressed the Bell inequalities in terms of conditional probabilities and suggested that all values of the hidden variables that gave the same output for every setting should be considered together. This led to much recent work including Froissart's [5] beautiful representation of the general inequalities as facets of polytopes, or many dimensional polyhedra [6, 7], and the use of transfer functions [8, 9] as in this paper. Conditional probabilities are better than expectation values because they are easier to generalize, and because they are obtained more directly from experiment.

Experiments of the Bell type, or simply *Bell experiments*, are designed to demonstrate weak nonlocality through the violation of some type of Bell inequality. None has yet done so without at least one loophole [10]. For example the experiment of Rowe *et al* [11] using entangled ions fails to overcome the locality or lightcone loophole by a factor of at least 10^{10}, and the entangled photon experiment of Weihs *et al* [12] fails for the detection loophole by a factor of about 17. Until an experiment succeeds in breaking a Bell inequality without any loophole, the possibility that we live in a completely local world cannot be ruled out [13].

There is an infinite number of possible Bell experiments, with entangled photons, ions or atoms, with entangled quantum systems of 2, 3 or more particles, and with different numbers of detectors with different efficiencies. For every one of them there is a specific set of Bell inequalities; the violation of any one of these inequalities without loopholes demonstrates weak nonlocality. If they are all satisfied, there is no such demonstration. These general Bell inequalities depend only on conditional probabilities of recorded classical events. There may be very many inequalities.

The computer program BellTest described here is freely available on the Web[1]. Given enough time, it generates all the Bell inequalities using Wigner's version of the inequalities [4]. In a second mode, which is usually much faster, it determines whether a given set of conditional probabilities satisfies all the inequalities or breaks at least one of them [14].

2. Bell experiments, inputs and outputs

There is a strong contrast between experiments designed to determine the properties of atoms, nuclei or particles on the one hand, and Bell experiments on the other. Properties such as spectra, lifetimes or cross-sections are quantum properties, to be compared either with field theory or with a solution of Schrödinger's equation. The classical apparatus is no more nor less than a means to this end. By contrast Bell experiments are designed to determine the probabilities of classical events, to be compared with the general theory of Bell inequalities, which is a purely classical theory of these events. The quantum system is no more nor less than a means to this end. The distinction is often ignored when interpreting the results of Bell experiments, leading to confusion about loopholes. The classical events are:

1. The classical inputs i that determine which property of the quantum system is to be measured. An example is the orientation of a polarizer.
2. The classical outputs j, which are the classical states of the data records.

Because the inequalities are between probabilities of classical events, the raw data should be used with sufficient statistics to estimate the conditional probabilities in the Bell inequalities, without any additional assumptions, post-selection or other tampering with these data. The conditional probabilities for the raw data *are* the results of the experiment. That is why the distinction between Bell experiments and other quantum experiments is so important. For the latter, processing of the data using the known physical properties of the apparatus is not only allowed, it is usually essential.

If the raw data are used and the spacetime conditions on the inputs and outputs are satisfied [8, 9], then the criteria for analysing the data and for improvement and eventual success of a Bell experiment with no loopholes are clear.

The *system* consists of the entire classical apparatus together with an entangled quantum system. The original Bell experiment with photons and the CHSH experiment contain two

[1] BellTest is freely available from http://www.strings.ph.qmw.ac.uk/QI/main.htm.

similar classical *subsystems*, A (Alice) and B (Bob), which are separated from one another at a minimum distance L. For a given run of the experiment, Alice sets the angle of a polarizing beam splitter at an angle φ_A and Bob sets his at an angle φ_B at the same time in the laboratory frame. These settings are the inputs. For the original Bell there are three possible settings for each and for CHSH there are two. In section 7 we consider more settings. For different runs the angles are chosen randomly from the possible settings. In an ideal CHSH experiment, it is assumed that Alice always detects one photon of an entangled photon pair, and Bob always detects the other, and that they are able to measure the states of polarization of each. The recording of these polarization states by Alice and Bob is the output, and it can be assumed that they occur at the same time. There are two possible outputs for Alice and two for Bob, making four in all. This assumption is not realistic for a laboratory experiment, since there is always a chance that one or both of the photons will not be detected. The CHSH theory takes account of this. There are only two detectors, one for Alice and one for Bob, that are supposed to detect only the parallel photons, and do not always succeed in doing that. Again there are only two possible outputs for Alice and for Bob, but these outputs do not correspond to the parallel and perpendicular polarization, as they would if the polarizers and detectors were perfect.

In order to test for nonlocality, it must not be possible to send a signal from Alice to Bob, or vice versa, during the experiment, so the maximum time T between the setting and recording process on both sides must satisfy the relativistic *locality* or *lightcone* condition $Tc < L$. Failure to do so results in the lightcone loophole, which is present for all Bell experiments to date that do not use entangled photons.

3. Transition and transfer pictures

We briefly introduce the general theory of input–output systems as it applies to deterministic systems and then to stochastic systems, such as Bell experiments. It is equivalent to the approach of Wigner [4] and Froissart [5]. A more detailed account is given in [8, 9].

Consider an input–output system composed from two independent input–output subsystems A and B. The system has input $i = (i_A, i_B)$ and output $j = (j_A, j_B)$, where $i_{A(B)}$ and $j_{A(B)}$ are the inputs and outputs of subsystem A (or B). Because we treat real experiments, we use CHSH as an example. Since there are two settings at A and B, there are four possible inputs i. Similarly, there are four possible outputs j, in which Alice's or Bob's detector does or does not fire.

For a given run with the input i there is a corresponding value of the output j. The relation between them is the transition $i \to j$. This is the *transition picture*. For the CHSH system there are 16 possible transitions. If A and B are so similar that they can be interchanged without changing the experiment, there is a symmetry which makes some of the transitions equivalent. Such symmetries help in the derivation of inequalities, but we do not consider them further in this paper.

In CHSH, the relation between the inputs and outputs is stochastic. But it is helpful to consider a *deterministic* system with the same inputs and outputs, which are linked classically so that the output is uniquely determined by the input. There is a unique j for each i, and the relation between them can be expressed as a *transfer function*

$$F(i_A, i_B) = (j_A, j_B). \tag{1}$$

This is the *transfer picture*. For CHSH the number of possible transfer functions is 256. There are local and nonlocal transfer functions. If j_B is independent of i_A, and j_A is independent of i_B, then the transfer function is local, and F can be expressed in terms of the transfer function for A and the transfer function for B. Otherwise it is nonlocal. A deterministic system with a nonlocal transfer function could be used to send a signal faster than the velocity of light. None has been found. For CHSH there are only 16 local transfer functions.

For stochastic systems the dynamics is defined by the probabilities of the transitions

$$\Pr(i \to j) = \Pr(j|i) \tag{2}$$

that is, the conditional probability of the output j given the input i. Because there is always *some* output, these sum to unity for every i:

$$\sum_j \Pr(j|i) = 1 \qquad \text{where} \quad 0 \leqslant \Pr(j|i) \leqslant 1. \tag{3}$$

Systems for which all the transition probabilities are either zero or one are deterministic.

In the transfer picture, the dynamics is defined by the probability that the system behaves like a deterministic system with transfer function F, for every possible F. These are transfer function probabilities, or transfer probabilities, $\Pr(F)$, which sum to unity. The conditional probability for finding the output j, when the input is i, is then given in terms of the transfer probabilities as

$$\Pr(j|i) = \sum_F \Pr(F)\delta(j, F(i)) \qquad \text{where} \quad \delta(j, F(i)) = \begin{cases} 0 & \text{if} \quad F(i) \neq j \\ 1 & \text{if} \quad F(i) = j. \end{cases} \tag{4}$$

This is the key equation, and gives the transition probabilities in terms of the transfer probabilities. The space of transfer probabilities is generally of much bigger dimension than the space of conditional probabilities. As a result, the key equation cannot usually be inverted to obtain the transfer probabilities in terms of the transition probabilities. However, inequalities for the transfer probabilities can be translated into equivalent inequalities for the transition probabilities using linear programming methods. Bell inequalities are an example.

In an experiment we measure the transition probabilities, but properties such as 'locality' are defined in terms of transfer probabilities. For given transition probabilities $\Pr(j|i)$, the probabilities $\Pr(F)$ must satisfy this key equation, together with the inequalities and normalization of the transfer probabilities, which are

$$0 \leqslant \Pr(F) \leqslant 1 \qquad \text{and} \qquad \sum_F \Pr(F) = 1. \tag{5}$$

For locality, only local transfer functions should be included in the sums of (4) and (5).

Using equation (4), the inequalities (5) can be translated into inequalities for the conditional probabilities $\Pr(j|i)$, which include all the Bell inequalities for the system. This is a general result which applies to experiments with any number of inputs and outputs. If, for measured $\Pr(j|i)$, a Bell inequality is violated, then at least one nonlocal transfer probability must be nonzero. Nevertheless, remarkably, according to quantum theory with ideal measurements, there are some systems which are nonlocal in this sense, and still cannot be used to send signals faster than the velocity of light. This is weak nonlocality, and forbids local hidden variable theories. It is a predicted property of some classical input and output events that are linked by entangled quantum systems. It has never been demonstrated, without loopholes, in a real experiment. This formulation, when applied to simple ideal experiments, is equivalent to the usual one in terms of hidden variables, but note that the Bell inequalities came from the classical spacetime relations of the inputs and the outputs, and the *formulation* needs no quantum mechanics at all. The role of the quantum system and quantum dynamics is to provide conditional probabilities that violate the inequalities, but it is not required in order to obtain the inequalities.

A similar formulation has been generalized to experiments with arbitrary numbers of inputs and outputs and of input and output values. It can also be applied to real experiments, that often have more values for outputs than ideal ones. An example is the non-detection of one of the photons of a photon pair. There are large numbers of inequalities of the Bell type, sometimes running into thousands for quite simple systems, which can be obtained systematically on a computer. However, for given experimental conditions, only one, or possibly a few, of these are relevant.

The classical events are the settings and the measurement outcomes. Weak nonlocality has not yet been demonstrated in real laboratory experiments, without further assumptions, or loopholes [10]. An experiment of the Bell type can be an effective test of weak nonlocality only if the runs are truly independent, the spacetime conditions are satisfied for all settings and measurement outcomes, and the statistics are adequate to provide sufficiently accurate values of the probabilities of the outcomes, given the settings.

4. Experiments without loopholes

There are inequalities for any number of apparatus settings and measurement outcomes, and even for any number of spatially separated subsystems (see for example [5, 15–20]). But as the number of settings, outcomes and subsystems increases, the number of inequalities increases very rapidly. For given experimental or theoretical transition probabilities, either all the inequalities are satisfied, in which case there is no demonstration of weak nonlocality, or at least one is violated, in which there *is* weak nonlocality.

Given these conditions, the transition probabilities for all the possible measurement outcomes have to be estimated. In planning an experiment this is done by using the properties of the apparatus and quantum dynamics to obtain the probabilities for an ideal experiment, and then correcting this with the estimated losses and sources of noise. This obviously involves trial and error.

No such estimates should be used for the *analysis* of an experiment of the Bell type, because the only reliable method of telling whether an experiment has demonstrated weak nonlocality is independent of any such estimates. Since real experiments often have more measurement outcomes than ideal experiments, the corresponding Bell inequalities are generally different.

From now on we suppose there are just two subsystems, as for CHSH, and that the lightcone condition is satisfied, and the number of settings is known. The number of detectors used on each side of the setup gives the the number of experimental outcomes.

5. Example

Experiments of the Bell type involve two or more entangled quantum systems, such as photons (e.g. [12, 21, 22]), or ions [11]. Consider the photon Bell experiment of Weihs *et al* [12] in which the locality loophole was closed. In this experiment, polarization entangled photon pairs, produced in a degenerate type-II parametric down-conversion process, were coupled to optical fibres that sent the photons to subsystems A and B which satisfied the lightcone condition. A and B each have one input and one output, and taken together form the system $A + B$ as shown in figure 1. Note that the state of the down converter which produces the entangled photons is the same for each run, and so does not appear in this analysis.

$A + B$ has classical input $i = (i_A, i_B)$ and classical output $j = (j_A, j_B)$. For the experiment of Weihs *et al* the outputs are the polarization measurement outcomes, and the

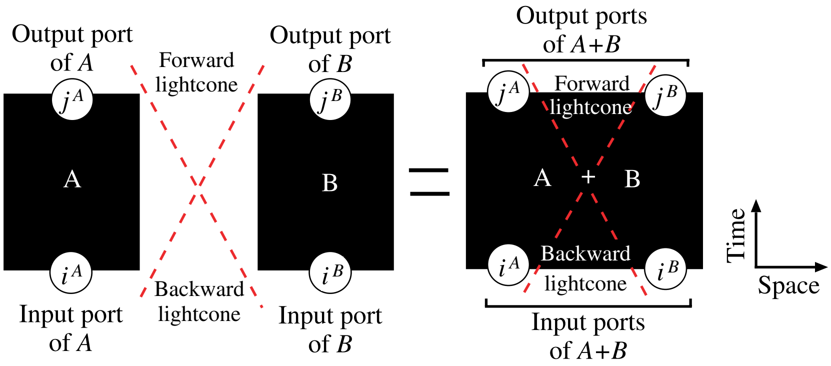

Figure 1. The above figure illustrates a black box system representing a Bell type experiment.

angle of the polarizer is the input. According to the classical (non-quantum) theory of special relativity, there can be no signalling between A and B, and so, only local transfer functions of the form

$$F = (F^A, F^B) \qquad \text{where} \quad \begin{cases} F^A(i_A i_B) = F^A(i_A) = j_A \\ F^B(i_A i_B) = F^B(i_B) = j_B \end{cases} \tag{6}$$

have nonzero probability $\Pr(F)$. Locality is equivalent to this condition, so the sum (4) is restricted to these terms. According to quantum dynamics and the quantum theory of ideal measurements, there are systems A and B (which are connected via an entangled quantum system) that violate this condition. At least one nonlocal transfer function may have to have nonzero probability. This is weak nonlocality [9], which is equivalent to Bell's original definition [2] in terms of hidden variables. Unfortunately, real experiments are not ideal, there are many possible loopholes and it is very difficult to demonstrate weak nonlocality in practice [10].

6. BellTest

For the analysis of Bell type experiments we make use of a computer program called *BellTest* [15], which is a well-tested and freely available program. It can be used in one of two possible modes:

Mode 1. Find the set of linear inequalities which put classical bounds on linear combinations of conditional probabilities (2).

Mode 2. Given all the conditional probabilities, perhaps obtained from the raw experimental data or from quantum mechanical predictions, test for weak nonlocality.

The program implements a simple connection between the general theory of input–output systems [8, 9] and the geometry of *convex polytopes* [6, 7]. A convex polytope, or simply polytope, is a bounded polyhedron of arbitrary dimension, described either in terms of a system of linear equalities and inequalities or by specifying every vertex. Both representations are equivalent and one can transform from one to the other. The polygons and the platonic solids are examples of two- and three-dimensional polytopes. The details underlying the relation between input–output systems and polytopes are not necessary for the results presented here and an account is given in a separate paper [23].

To use BellTest, the Bell experiment must first be described in terms of inputs and outputs. This amounts to specifying the total number of subsystems along with the number of inputs and outputs per subsystem. To test weak nonlocality, the inputs and outputs of every subsystem must be space-like separated from the inputs and outputs of every other subsystem.

For a given Bell experiment, the corresponding set of conditional $\Pr(j|i)$ probabilities taken together define a point in a space with Cartesian coordinates, where i and j are strings of inputs and outputs over all input and output ports. Moreover, equation (4), relating the transition and transfer pictures, and constraints (3) and (5), for all inputs and outputs, define a subspace of *allowed* transition probabilities, the *Bell polytope*, whose vertices represent local transfer functions F with the qualification $\Pr(F) \neq 0$. These are determined by BellTest and Mode 1 is nothing more than the *facet enumeration* problem, where, given all the vertices, one is interested in obtaining the set of linear inequalities and equalities that define the same convex polytope. The facet enumeration problem has attracted much attention from computational geometers and as a result an extensive library of computational tools is readily available over the Web. We make use of the software called *Polyhedron Representation Transformation Algorithm* [24] to list all the Bell inequalities for a given set of local transfer functions with non-zero probability.

In general, the number of local transfer functions grows exponentially, and what is more the number of inequalities that must be satisfied also grows in an exponential manner. In addition, even for small numbers of subsystems, inputs and outputs, the time taken to run BellTest in Mode 1 can be large. To this end, given all the transition probabilities, checking every inequality and equality for a violation is not the most efficient test for weak nonlocality as Mode 1 must be carried out first. However, at least one nonlocal transfer function with non-zero probability is required to satisfy (4) for all inputs and outputs whenever an inequality or equality is violated. In Mode 2, BellTest produces a Mathematica notebook which contains a linear program. However, we are not interested in obtaining the maximum or minimum of a linear expression with respect to a set of constraints, but rather, we only want to know if the program has a feasible solution. The variable quantities are the local transfer probabilities $\Pr(F)$, subject to equalities (4) for all inputs and outputs. We make use of Mathematica's linear programming package for solving the feasibility problem[2].

Pitowsky and Svozil [25] and Filipp and Svozil [26] have produced programs which operate in a manner similar to that of BellTest in Mode 1. Kaszlikowski and Zukowski [29] have also produced a program which operates similarly to BellTest in Mode 2.

7. General CHSH inequalities

The experiment proposed by CHSH involves two settings of the polarizers at either site; that is, two inputs i_A at A and two inputs i_B at B. Generalizations of the CHSH experiment to experiments using more than two inputs either at A or at B for two entangled two-state systems have been considered by other authors and a number of generalized CHSH inequalities have appeared in the literature [17, 27–34]. We have also considered such experiments. In particular, we have applied BellTest to two types of generalized CHSH experiments with $N'(i)$ inputs at either end.

E1. Experiments that make use of $2N'(i)$ out of the $N'(i)^2$ possible inputs $i = (i_A i_B)$.
E2. Experiments that use all $N'(i)^2$ experimental inputs $i = (i_A i_B)$.

However, due to the rapid growth in the computational time taken to run BellTest in either mode as the number of inputs increases, we have only considered cases involving small $N'(i)$. Experiments of type E1 were first considered in [28] and later in [30, 31], and the

[2] Mathematica is a commercial computational tool. For more information please go to http://www.wolfram.com/

corresponding set of Bell inequalities were called the *chained Bell inequalities*, which are obtained by taking linear combinations of CHSH inequalities for appropriate pairs of inputs. In [34], Gisin presented a generalized CHSH inequality for experiments of type E2, for which as $N'(i)$ increases the ratio of violation approaches the limit $4/\pi$. However, Peres [17] has conjectured that for this class of experiments it is sufficient to consider CHSH inequalities for pairs of inputs at A and B. The computational results obtained by Kaszlikowski *et al* (see chapter 8 of [27]) show, for the cases $2 \leqslant N'(i) \leqslant 10$, that the critical visibility required to satisfy a local hidden variable model reproducing the quantum predictions cannot be greater than $1/\sqrt{2}$, which is the critical visibility to violate the CHSH inequality. Their computational results clearly bolster the conjecture put forward by Peres.

First we used BellTest in Mode 1 to obtain, for specific cases, all the inequalities that must be satisfied by the conditional probabilities $\Pr(j_A j_B|i_A i_B)$ obtained for both types of generalized CHSH experiments. For experiments of type E1, we have obtained all the inequalities for $N'(i) = 3, 4, 5$, and for experiments of type E2, we have obtained all the inequalities for $N'(i) = 2, 3, 4$. As the number of inequalities increases rapidly with respect to the number of inputs at either end (this is especially so for experiments of type E2), it is not possible to list them all here, and consequently, we only give examples of the different types of inequalities. The full set can be viewed on the website. For cases $2 \leqslant N'(i) \leqslant 7$ we have also applied BellTest in Mode 2 to experiments of type 2, involving entangled pairs of two-state quantum systems in the well-known mixed state

$$\rho = \lambda\rho_{\text{noise}} + (1-\lambda)\rho_{\text{max}} \tag{7}$$

where

$$\rho_{\text{noise}} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} \qquad \rho_{\text{max}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

and $0 \leqslant \lambda \leqslant 1$ describes the fraction of noise present in ρ. For simplicity, throughout our analysis, we have assumed ideal preparation and measurement. Therefore, there are two possible output events at A and B for any given run of experiments E1 and E2. We denote these + and −, which represent detection and non-detection of a photon, respectively. There are four possible outputs $j = (j_A j_B) \in \{(++), (+-), (-+), (--)\}$ given the input event $i = (i_A i_B)$. To keep down the time taken to run BellTest in either mode, we have made use of the transfer probability symmetries. For Mode 1, all the inequalities found will be expressions involving the conditional probability measures $\Pr(++|i_A i_B)$ only. However, the inequalities can be expressed in terms of expectations $\mathrm{E}(i_A i_B)$ through the relation

$$\mathrm{E}(i_A i_B) = 4\Pr(++|i_A i_B) - 1 \tag{9}$$

where we have made use of the symmetries:

$$\Pr(++|i_A i_B) = \Pr(--|i_A i_B) \qquad \Pr(+-|i_A i_B) = \Pr(-+|i_A i_B) \tag{10}$$

and constraint (3).

The smallest chained CHSH experiment involves $N'(i) = 3$ inputs at both A and B. BellTest labels the inputs and outputs in terms of integers, and for this case the inputs i_A and i_B can be any value from the integer set {0, 1, 2}. Similarly, the outputs j_A and j_B can be any value from the set {0, 1}, and we may take the output value 0 to represent − and 1 to represent +. The set of all possible inputs $i = (i_A i_B)$ is therefore

$$\begin{matrix} 0\ 0 & 0\ 1 & 0\ 2 \\ 1\ 0 & 1\ 1 & 1\ 2 \\ 2\ 0 & 2\ 1 & 2\ 2. \end{matrix} \tag{11}$$

However, for this chained CHSH experiment only six inputs are used, namely

$$0\ 0,\quad 0\ 2,\quad 1\ 0,\quad 1\ 1,\quad 2\ 1,\quad 2\ 2. \tag{12}$$

For this chained CHSH experiment BellTest finds that there are 44 inequalities that must be satisfied by the conditional probabilities. The inequalities found were of three different types:

$$\Pr(++|i_A i_B) \geqslant 0 \qquad 2\Pr(++|i_A i_B) \leqslant 1 \qquad \text{for all inputs } i_A i_B \tag{13}$$

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &-\Pr(++|a_3b_2) - \Pr(++|a_3b_3) - \Pr(++|a_1b_3) \leqslant 1\end{aligned} \tag{14}$$

$$\begin{aligned}0 \leqslant &+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) - \Pr(++|a_1b_3) \leqslant 2\end{aligned} \tag{15}$$

where a_k and b_l represent the different settings of A and B. Inequalities (13) are trivial. Inequalities (15) are chained Bell inequalities which, by using relation (9), can be written in their standard form [31]. However inequalities (14) appear to be new. For appropriate settings, the conditional probabilities as predicted by quantum mechanics can be shown to violate (14) and (15). In a similar manner, we also found all the inequalities for chained CHSH experiments with $N'(i) = 4$ and 5. For $N'(i) = 4$, BellTest obtained non-trivial inequalities of the form

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &-\Pr(++|a_3b_2) - \Pr(++|a_3b_3) - \Pr(++|a_4b_3)\\ &\qquad - \Pr(++|a_4b_4) - \Pr(++|a_1b_4) \leqslant 1\end{aligned} \tag{16}$$

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) - \Pr(++|a_4b_3)\\ &\qquad - \Pr(++|a_4b_4) - \Pr(++|a_1b_4) \leqslant 2\end{aligned} \tag{17}$$

$$\begin{aligned}0 \leqslant &+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) + \Pr(++|a_4b_3)\\ &\qquad + \Pr(++|a_4b_4) - \Pr(++|a_1b_4) \leqslant 3.\end{aligned} \tag{18}$$

For $N'(i) = 5$, the following set of non-trivial inequality types were obtained:

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &-\Pr(++|a_3b_2) - \Pr(++|a_3b_3) - \Pr(++|a_4b_3)\\ &-\Pr(++|a_4b_4) - \Pr(++|a_5b_4) - \Pr(++|a_5b_5)\\ &\qquad - \Pr(++|a_5b_4) \leqslant 1\end{aligned} \tag{19}$$

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) - \Pr(++|a_4b_3)\\ &-\Pr(++|a_4b_4) - \Pr(++|a_5b_4) - \Pr(++|a_5b_5)\\ &\qquad - \Pr(++|a_5b_4) \leqslant 2\end{aligned} \tag{20}$$

$$\begin{aligned}&+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) + \Pr(++|a_4b_3)\\ &+\Pr(++|a_4b_4) - \Pr(++|a_5b_4) - \Pr(++|a_5b_5)\\ &\qquad - \Pr(++|a_5b_4) \leqslant 3\end{aligned} \tag{21}$$

$$\begin{aligned}0 \leqslant &+\Pr(++|a_1b_1) + \Pr(++|a_2b_1) + \Pr(++|a_2b_2)\\ &+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) + \Pr(++|a_4b_3)\\ &+\Pr(++|a_4b_4) + \Pr(++|a_5b_4) + \Pr(++|a_5b_5)\\ &\qquad - \Pr(++|a_5b_4) \leqslant 4.\end{aligned} \tag{22}$$

Inequalities (18) and (22) are chained Bell inequalities for $N'(i) = 4$ and 5, respectively. The other inequalities, which like inequalities (14) appear to be new, can be violated for appropriate choices of inputs.

Let us now consider experiments of the type E2. For $N'(i) = 2$, BellTest obtains only the trivial inequalities of the form (13) and the CHSH inequalities:

$$0 \leqslant \Pr(++|a_1b_1) + \Pr(++|a_1b_2) + \Pr(++|a_2b_1) - \Pr(++|a_2b_2) \leqslant 1. \quad (23)$$

For $N'(i) = 3$ the non-trivial inequalities are all of the CHSH type, and could have been obtained without BellTest by selecting the inputs in pairs from A and from B and writing the CHSH inequalities for these pairs. There is no need for the computer program, except to ensure that there are no other inequalities. A natural extrapolation might suggest that the same is true for all values of $N'(i)$, but, surprisingly, this is not so. For $N'(i) = 4$, there are inequalities of a completely new type. BellTest found the following set of non-trivial and non-CHSH type inequalities:

$$\begin{aligned}
&-\Pr(++|a_1b_1) - \Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_1b_4)\\
&-\Pr(++|a_2b_1) - \Pr(++|a_2b_2) - \Pr(++|a_2b_3) + \Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) - \Pr(++|a_3b_2) + 2\Pr(++|a_3b_3) - \Pr(++|a_4b_1)\\
&\qquad + \Pr(++|a_4b_2) \leqslant 0
\end{aligned} \quad (24)$$

$$\begin{aligned}
&-2\Pr(++|a_1b_1) - \Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_2b_1)\\
&+\Pr(++|a_2b_2) + \Pr(++|a_2b_3) - \Pr(++|a_2b_4) - \Pr(++|a_3b_2)\\
&+\Pr(++|a_3b_3) + \Pr(++|a_4b_1) - \Pr(++|a_4b_2) - \Pr(++|a_4b_3)\\
&\qquad - \Pr(++|a_4b_4) \leqslant 0
\end{aligned} \quad (25)$$

$$\begin{aligned}
&-2\Pr(++|a_1b_1) - 2\Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_1b_4)\\
&-\Pr(++|a_2b_1) + \Pr(++|a_2b_2) - 2\Pr(++|a_2b_3) + 2\Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) - \Pr(++|a_3b_3) - 2\Pr(++|a_3b_4)\\
&+2\Pr(++|a_4b_1) - \Pr(++|a_4b_2) - 2\Pr(++|a_4b_3) - \Pr(++|a_4b_1) \leqslant 0
\end{aligned} \quad (26)$$

$$\begin{aligned}
&-2\Pr(++|a_1b_1) - 2\Pr(++|a_1b_2) - \Pr(++|a_1b_3) + \Pr(++|a_1b_4)\\
&-2\Pr(++|a_2b_1) + \Pr(++|a_2b_2) + \Pr(++|a_2b_3) - 2\Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) - 2\Pr(++|a_3b_3) + \Pr(++|a_3b_4)\\
&+\Pr(++|a_4b_1) - \Pr(++|a_4b_2) - 2\Pr(++|a_4b_1) - 2\Pr(++|a_4b_1) \leqslant 0
\end{aligned} \quad (27)$$

$$\begin{aligned}
&-\Pr(++|a_1b_1) - \Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_1b_4)\\
&-\Pr(++|a_2b_1) - \Pr(++|a_2b_2) + 2\Pr(++|a_2b_4) - \Pr(++|a_3b_1)\\
&+\Pr(++|a_3b_2) + \Pr(++|a_4b_1) + \Pr(++|a_4b_2) - \Pr(++|a_4b_3)\\
&\qquad + \Pr(++|a_4b_4) \leqslant 1
\end{aligned} \quad (28)$$

$$\begin{aligned}
&-2\Pr(++|a_1b_1) - \Pr(++|a_1b_2) - \Pr(++|a_1b_1) - \Pr(++|a_2b_1)\\
&+\Pr(++|a_2b_2) + \Pr(++|a_2b_3) - \Pr(++|a_2b_4) - \Pr(++|a_3b_1)\\
&+\Pr(++|a_3b_2) + \Pr(++|a_3b_3) + \Pr(++|a_3b_4) - \Pr(++|a_4b_2)\\
&\qquad + \Pr(++|a_4b_3) \leqslant 1
\end{aligned} \quad (29)$$

$$\begin{aligned}
&-2\Pr(++|a_1b_1) - 2\Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_1b_4)\\
&-2\Pr(++|a_2b_1) + \Pr(++|a_2b_2) + \Pr(++|a_2b_3) + 2\Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + \Pr(++|a_3b_2) + 2\Pr(++|a_3b_3) - 2\Pr(++|a_3b_4)\\
&-\Pr(++|a_4b_1) + 2\Pr(++|a_4b_2) - 2\Pr(++|a_4b_3) - \Pr(++|a_4b_4) \leqslant 1
\end{aligned} \quad (30)$$

$$
\begin{aligned}
&-\Pr(++|a_1b_1) - \Pr(++|a_1b_2) - \Pr(++|a_1b_3) + \Pr(++|a_1b_4)\\
&-\Pr(++|a_2b_1) - \Pr(++|a_2b_2) + \Pr(++|a_2b_3) + \Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + \Pr(++|a_3b_2) + \Pr(++|a_4b_1) + \Pr(++|a_4b_2)\\
&\qquad\qquad\qquad\qquad +2\Pr(++|a_4b_4) \leqslant 2
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) - \Pr(++|a_1b_2) + \Pr(++|a_1b_4) - \Pr(++|a_2b_1)\\
&+\Pr(++|a_2b_2) + \Pr(++|a_2b_3) - \Pr(++|a_2b_4) + \Pr(++|a_3b_2)\\
&+\Pr(++|a_3b_4) + \Pr(++|a_4b_1) - \Pr(++|a_4b_2) + \Pr(++|a_4b_3)\\
&\qquad\qquad\qquad\qquad +\Pr(++|a_4b_4) \leqslant 2
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) - 2\Pr(++|a_1b_2) - \Pr(++|a_1b_3) - \Pr(++|a_1b_4)\\
&-2\Pr(++|a_2b_1) + \Pr(++|a_2b_2) + \Pr(++|a_2b_3) + 2\Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + \Pr(++|a_3b_2) + 2\Pr(++|a_3b_3) - 2\Pr(++|a_3b_4)\\
&+\Pr(++|a_4b_1) - 2\Pr(++|a_4b_2) + 2\Pr(++|a_4b_3) + \Pr(++|a_4b_4) \leqslant 2
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&-\Pr(++|a_1b_1) - \Pr(++|a_1b_2) + 2\Pr(++|a_1b_4) - \Pr(++|a_2b_1)\\
&+\Pr(++|a_2b_2) + \Pr(++|a_3b_1) + \Pr(++|a_3b_2) - \Pr(++|a_3b_3)\\
&+\Pr(++|a_3b_4) + \Pr(++|a_4b_1) + \Pr(++|a_4b_2) + \Pr(++|a_4b_3)\\
&\qquad\qquad\qquad\qquad +\Pr(++|a_4b_4) \leqslant 3
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) + \Pr(++|a_1b_3) + \Pr(++|a_1b_4) - \Pr(++|a_2b_3)\\
&+\Pr(++|a_2b_4) + \Pr(++|a_3b_1) - \Pr(++|a_3b_2) + \Pr(++|a_3b_3)\\
&+\Pr(++|a_3b_4) + \Pr(++|a_4b_1) + \Pr(++|a_4b_2) + \Pr(++|a_4b_3)\\
&\qquad\qquad\qquad\qquad +\Pr(++|a_4b_4) \leqslant 3
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) - 2\Pr(++|a_1b_2) - \Pr(++|a_1b_3) + \Pr(++|a_1b_4)\\
&-2\Pr(++|a_2b_1) + \Pr(++|a_2b_2) + 2\Pr(++|a_2b_3) - \Pr(++|a_2b_4)\\
&-\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) - \Pr(++|a_3b_3) + 2\Pr(++|a_3b_4)\\
&+\Pr(++|a_4b_1) - \Pr(++|a_4b_2) + 2\Pr(++|a_4b_3) + 2\Pr(++|a_4b_4) \leqslant 3
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&+2\Pr(++|a_1b_1) + \Pr(++|a_1b_2) + \Pr(++|a_1b_3) + 2\Pr(++|a_1b_4)\\
&+\Pr(++|a_2b_1) + \Pr(++|a_2b_2) + 2\Pr(++|a_2b_3) - 2\Pr(++|a_2b_4)\\
&+\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) - 2\Pr(++|a_3b_3) - \Pr(++|a_3b_4)\\
&+2\Pr(++|a_4b_1) - 2\Pr(++|a_4b_2) - \Pr(++|a_4b_3) - \Pr(++|a_4b_4) \leqslant 4
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) - \Pr(++|a_1b_2) + \Pr(++|a_1b_3) + 2\Pr(++|a_1b_4)\\
&-\Pr(++|a_2b_1) + 2\Pr(++|a_2b_2) + 2\Pr(++|a_2b_3) - \Pr(++|a_2b_4)\\
&+\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) - \Pr(++|a_3b_3) + 2\Pr(++|a_3b_4)\\
&+2\Pr(++|a_4b_1) - \Pr(++|a_4b_2) + 2\Pr(++|a_4b_3) + \Pr(++|a_4b_4) \leqslant 5
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&-2\Pr(++|a_1b_1) - \Pr(++|a_1b_2) + \Pr(++|a_1b_3) + 2\Pr(++|a_1b_4)\\
&+\Pr(++|a_2b_1) + \Pr(++|a_2b_2) - 2\Pr(++|a_2b_3) + 2\Pr(++|a_2b_4)\\
&+\Pr(++|a_3b_1) + 2\Pr(++|a_3b_2) + 2\Pr(++|a_3b_3) + \Pr(++|a_3b_4)\\
&+2\Pr(++|a_4b_1) - 2\Pr(++|a_4b_2) + \Pr(++|a_4b_3) + \Pr(++|a_4b_4) \leqslant 5.
\end{aligned} \tag{39}
$$

Inequalities (24) to (39), for appropriate choices of inputs, can be violated by quantum mechanics.

For the cases studied, we have seen, for experiments of type E2, that the simplest non-trivial inequality is always a CHSH type inequality. For $N'(i) > 2$, by taking linear combinations of appropriate CHSH type inequalities not only can we obtain the chained Bell inequalities (15), (18) and (22) (as expected), but also the other non-trivial inequalities. For example, in the case $N'(i) = 3$, the chained Bell inequality (15) can be obtained by adding the following CHSH type inequalities obtained for the corresponding E2 type experiment:

$$\begin{aligned} &\Pr(++|a_1b_1) - \Pr(++|a_1b_3) + \Pr(++|a_3b_1) + \Pr(++|a_3b_3) \leqslant 1 \\ &\Pr(++|a_2b_1) + \Pr(++|a_2b_2) - \Pr(++|a_3b_1) + \Pr(++|a_3b_2) \leqslant 1. \end{aligned} \tag{40}$$

Likewise, inequality (14) can be obtained as a linear combination of inequalities

$$\begin{aligned} &\Pr(++|a_1b_1) - \Pr(++|a_1b_3) + \Pr(++|a_2b_1) + \Pr(++|a_2b_3) \leqslant 1 \\ &\Pr(++|a_2b_2) - \Pr(++|a_2b_3) - \Pr(++|a_3b_2) - \Pr(++|a_3b_3) \leqslant 0. \end{aligned} \tag{41}$$

The time taken for BellTest to find all the inequalities can, in general, be large, and consequently the range of experiments that can be analysed in Mode 1 is small. Running BellTest in Mode 2 does not require use of the inequalities explicitly. Instead, for all inputs and outputs, the conditional probabilities must be provided by the user. More importantly, in Mode 2, a greater range of experiments can be analysed. We considered experiments of type E2 with $2 \leqslant N'(i) \leqslant 7$, and for appropriate inputs we determined the critical noise λ' needed to express the quantum mechanical predictions for the conditional probabilities satisfying state (9) in terms of local transfer probabilities $\Pr(F) \neq 0$. For each case, the feasibility test, a Mathematica notebook generated by BellTest, was iterated for different values of λ using a bisection method until $\lambda = \lambda'$ (to five decimal places). Using only those input events that maximally violate Gisin's generalized CHSH inequality [34], we found that the critical noise, λ'_{BT}, required for the quantum predictions to satisfy equation (4) is always greater than the critical noise, λ'_{GI}, required to satisfy Gisin's inequality. Consequently, for all inputs and outputs, and for the range of experiments considered, Gisin's generalized CHSH inequality gives weaker constraints on λ' than the constraints obtained from (4). However, for other choices of inputs, we found that for $3 \leqslant N'(i) \leqslant 7$ there is a maximum critical noise of $1 - 1/\sqrt{2}$, which is the critical noise required to violate the CHSH inequality. This result lends support to the findings of Kaszlikowski [27] and Peres's conjecture [17].

8. Conclusion

BellTest is a generally available and well-tested computer program for obtaining inequalities of the Bell type and for testing whether the raw data of a Bell experiment satisfy its appropriate set of inequalities. It can be used to help in the preliminary stages of the design of experiments of the Bell type and in analysing the final results.

We have applied BellTest to the well-known problem of CHSH experiments, for two entangled two state quantum systems, that utilize more than two inputs (apparatus settings) at either end of the setup. In particular, given $N'(i)$ inputs at A and B, we considered two such generalizations of the CHSH experiment: experiments that use only $2N'(i)$ of the possible $N'(i)^2$ inputs and experiments that use all possible inputs. In Mode 1, we obtained some old and new inequalities. In Mode 2, we considered a special example of generalized CHSH experiments that use all $N'(i)^2$ inputs, and the results obtained lend support to the findings of Peres [17] and Kaszlikowski [27].

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